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Some Intrinsic Properties of Steady Parallel Flows in Magneto-Fluid Dynamics

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1. INTRODUCTION

In this paper three-dimensional steady parallel flows of inviscid magnetic fluids are considered. Parallel flows have the property that the fluid velocity and magnetic field are locally parallel. Here the geometry of such flows are investigated.

Grad [1] has shown that an incompressible parallel flow is an exact mathematical analog of a problem in nonmagnetic fluid dynamics; Germain [2] has also discussed such flows. Smith [3] has studied parallel flows in the compressible case and has generalized some properties of the nonmagnetic flows by the use of substitution principle [4]. In this paper some of the geometric properties of the fluid flows are extended to parallel flows. Considering a streamline as a C^3 curve in E^3, a Euclidean three-dimensional space, the intrinsic forms of the equations governing the flow are derived. It is found that the hydromagnetic pressure does not vary along the binormal and that its variation along the principal normal depends on the curvature of the streamline. This result holds for fluid pressure in nonmagnetic fluid dynamics [5], and for the hydromagnetic pressure for the flows in which the magnetic field is along a fixed direction [6]. It is shown that for a gas admitting a separable equation of state, the normals to the surfaces of constant pressure, entropy, density, and velocity magnitude—these quantities being reduced in the sense given by the substitution principle—are coplanar at every point of the flow and that the cross-ratio of the pencil of these lines through a point is constant along the surfaces of constant pressure. In particular if the gas is polytropic, the cross ratio is constant throughout the medium of the flow and is equal to the adiabatic exponent. This result is a direct generalization of the theorems proved by Smith [7] for nonmagnetic gas flows. For nonmagnetic barotropic flows [8], it is known that the Bernoulli surfaces contain both the streamlines and the vortex lines ([9], p. 405), but in the case of parallel flows the Bernoulli surfaces in general contain the streamlines only.

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In the case of parallel flows, it is found that if the divergence of the unit tangent vector along the velocity vanishes, then the flow of the fluid has either sonic speed or its pressure along the streamlines constant. The same result holds for ordinary fluid flow [10]. The basic equations are obtained for a subclass of these flows, the class of all parallel helical flows. It is found that the hydromagnetic pressure depends only on the radius of the cylinder. Wasserman [11] has shown, for nonmagnetic fluid flows, that the pressure is an nondecreasing function of the radius of the cylinder; but for parallel flows both the pressure and the hydromagnetic pressure are not in general increasing functions of the radius of the cylinder. An example of parallel helical flows is given for which both the pressure and the hydromagnetic pressure are decreasing functions of the radius. Flows of nonmagnetic fluids for which the streamlines are geodesics on surfaces of revolutions (note, a helix is a geodesic on a right circular cylinder) are discussed by Wasserman [12].

2. The Equations

Let x^j ($j = 1, 2, 3$) denote the variables of a system of Cartesian orthogonal coordinates. In order to apply the summation convention, we shall use the indices in covariant and contravariant positions. We shall write $\partial_i = \partial/\partial x^i$. By use of the magneto-hydrodynamical approximation, the equations of continuity and motion and the Maxwell relations for a inviscid, infinitely conductive, compressible or incompressible fluid become [13],

$$\partial_i(\rho u^i) = 0, \tag{2.1}$$

$$\rho u^j \, \partial_j u_i + \partial_i(p + \tfrac{1}{2}\,\mu H^2) - \mu H^j \, \partial_j H_i = 0, \tag{2.2}$$

$$u^j \, \partial_j H_i - H^j \, \partial_j u_i + H_i \, \partial_j u^j = 0, \tag{2.3}$$

$$\partial_i H^i = 0, \tag{2.4}$$

$$u^j \, \partial_j \eta = 0, \qquad p = f(\rho, \eta), \tag{2.5}$$

where μ is the magnetic permiability; p is the fluid pressure, ρ the density, η the entropy, H is the magnitude of the magnetic field H^i; u^i are the components of the velocity. For a Prim gas, the equation of state is separable and is given by

$$\rho = P(p)\, S(\eta). \tag{2.6}$$

For a polytropic gas $P(p) = p^{1/\gamma}$, where γ is the adiabatic exponent.

3. Parallel Flows

For parallel flows u^i is parallel to H^i and therefore

$$H^i = \lambda u^i, \tag{3.1}$$

where λ is a scalar. Substituting for H^i from (3.1) into (2.4) and comparing with (2.1) we find that

$$\lambda = \alpha\rho, \qquad u^j\, \partial_j\alpha = 0, \tag{3.2}$$

where α is a scalar. From (3.1) and (3.2) we find that

$$H^i = \alpha\rho u^i. \tag{3.3}$$

By substituting for H^i from (3.3) into (2.2)-(2.4), the Eqs. (2.1)-(2.5) governing the motion for parallel flows in Magneto-fluid-dynamics now become

$$\partial_i(\rho u^i) = 0, \tag{3.4}$$

$$\rho(1 - \mu\alpha^2\rho)\, u^j\, \partial_j u_i - \mu\alpha\rho u^j\, \partial_j(\alpha\rho)\, u_i + \partial_i p + \tfrac{1}{2}\,\mu\, \partial_i(\alpha\rho u)^2 = 0, \tag{3.5}$$

$$u^j\, \partial_j\alpha = u^j\, \partial_j\eta = 0, \tag{3.6}$$

$$p = f(\rho, \eta), \tag{3.7}$$

where u is the magnitude of u^i. This is a system of seven equations in seven unknowns u^i, ρ, p, η, α.

4. The Basic Decomposition

If s^i denotes the unit tangent vector along the streamline, we may write

$$u^i = us^i. \tag{4.1}$$

By substituting for u^i from (4.1) into (3.4)-(3.6), these equations become

$$\frac{1}{\rho u}\frac{d(\rho u)}{ds} + \bar{M} = 0 \tag{4.2}$$

$$\left[\rho u(1 - \mu\rho\alpha^2)\frac{du}{ds} - \mu\rho u^2\alpha\frac{d}{ds}(\alpha\rho)\right] s_i + \rho(1 - \mu\alpha^2\rho)\, u^2 k n_i + \partial_i \Pi = 0, \tag{4.3}$$

$$\frac{d\alpha}{ds} = 0 = \frac{d\eta}{ds}, \tag{4.4}$$

where $\bar{M}$ is the divergence of the vector s_i and k is the curvature of the streamline, n_i is the unit principal normal vector of the streamline, d/ds denotes the directional derivative operator along the streamline and

$$\Pi = p + \tfrac{1}{2}\,\mu(\alpha\rho u)^2 \tag{4.5}$$

is the hydromagnetic pressure. Taking the scalar product of (4.3) with s^i, n^i, and b^i, the unit binormal vector of the streamline and making use of their orthonormal conditions we find that

$$\rho(1-\mu\rho\alpha^2)\,u\frac{du}{ds}-\mu\alpha^2\rho u^2\frac{d\rho}{ds}+\frac{d\Pi}{ds}=0 \tag{4.6}$$

$$\rho(1-\mu\rho\alpha^2)\,u^2k+\frac{d\Pi}{dn}=0, \tag{4.7}$$

$$\frac{d\Pi}{db}=0, \tag{4.8}$$

where d/dn and d/db are the directional derivative operators along the principal normal and the binormal vectors respectively. *The equations* (4.2), (4.4), (4.6)-(4.8) *are the intrinsic formulations of the equation* (3.4)-(3.6). Relations (4.7) and (4.8) imply that the *hydromagnetic pressure Π does not vary along the binormal vector and its variation along the principal normal vector depends on the curvature of the streamline.* The Eqs. (4.7) and (4.8) hold in the three-dimensional fluid flows in the nonmagnetic case if Π is replaced by p [5]. They also hold for hydromagnetic fluid flows, when the magnetic field has a fixed direction [6]. From (4.7) we have that in the case of straight line flows the hydromagnetic pressure does not vary along the principal normal and conversely.

By substituting for Π from (4.5) into (4.6) and cancelling terms, the equation (4.6) reduces to (cf. p. 836 [1], p. 14 [2]).

$$\frac{d}{ds}\left(\frac{1}{2}u^2\right)+\frac{1}{\rho}\frac{dp}{ds}=0. \tag{4.9}$$

By substituting for ρ from (2.6), the above equation becomes

$$\frac{d}{ds}\left(\frac{1}{2}u^2\right)+\frac{1}{S(\eta)\,P(p)}\frac{dp}{ds}=0.$$

This equation can be written in the form

$$u^2+2\frac{G(p)}{S(\eta)}=q^2, \tag{4.10}$$

where q is the ultimate velocity magnitude and

$$G(p)=\int_0^p\frac{dx}{P(x)}\,.$$

We now generalize a result due to Smith [7], for a Prim gas. Using the substitution principle [4], we can define a new velocity vector v^i, density σ and entropy ζ by

$$qv^i = u^i, \qquad \sigma = \rho q^2, \qquad S(\zeta) = q^2 S(\eta). \tag{4.11}$$

By use of (4.1), the equation (4.10) becomes

$$v^2 + 2\frac{G(p)}{S(\zeta)} = 1, \tag{4.12}$$

where v is the magnitude of the reduced velocity vector v^i. The equation of state (2.6), by use of (4.11), becomes

$$\sigma = P(p)\, S(\zeta). \tag{4.13}$$

The equations (4.12), (4.13) are precisely the equations used by Smith [7] to prove that the normals to the surfaces of constant p, ζ, σ, v are coplanar and that the cross-ratio of the pencil of lines formed by these normals is constant along the surfaces of constant pressure and that the cross-ratio is equal to γ if the gas is polytropic (γ being the adiabatic exponent). Therefore we have the following results.

In the parallel flow of an inviscid magnetic Prim gas, the pencil of lines formed by the normals to the surfaces of constant p, ζ, σ and v is coplanar and the cross-ratio of the pencil is constant along the surface of constant pressure.

In the parallel flow of a magnetic polytropic gas, the cross-ratio of the pencil of lines formed by the normals to the surfaces of constant p, ζ, σ and v is constant throughout the flow and is equal to the adiabatic exponent.

The converse of the second result is also true, that is *the cross-ratio is only constant for a polytropic gas* [7].

The pencil will be harmonic for $\gamma = -1$ which corresponds to the Chaplygin gas.

5. A Substitute Flow

Since α is constant along a streamline, let us consider the following transformation

$$\alpha v^i = u^i, \qquad \sigma = \rho\alpha^2, \qquad p = p, \qquad \zeta = \zeta(\alpha, \eta), \tag{5.1}$$

where q of (4.1) is replaced by α of (3.2). Note, α is a C^1 function of η and ξ, the two independent integrals of the first equation of (2.5). Hence v^i, σ and ζ are now different from those appearing in the Section 4. Substituting for u^i, ρ

and η from (5.1) into (3.4)-(3.7), the relations (3.4)-(3.7) become (since α is constant along any streamline)

$$\partial_i(\sigma v^i) = 0,$$
$$\sigma(1 - \mu\sigma)\, v^j\, \partial_j v_i - \mu\sigma(v^j\, \partial_j \sigma)\, v_i + \partial_i p + \tfrac{1}{2}\,\mu\, \partial_i(\sigma v)^2 = 0,$$
$$v^j\, \partial_j \zeta = 0, \qquad p = p(\sigma, \zeta).$$

The above equations hold for the substitute flow F^* of the original flow F given by (3.4)-(3.7). Both the flows have the same streamline pattern. The equations for F^* consists of six equations in six variables to be determined v^i, p, σ, ζ. Substituting for u^i, ρ, η from (5.1) into (4.2), (4.6)-(4.8), we find that

$$\frac{1}{\sigma v}\frac{d}{ds}(\sigma v) + \bar{M} = 0 \tag{5.2}$$

$$\sigma(1 - \mu\sigma)\, v \frac{dv}{ds} + \frac{d\Pi}{ds} = 0 \tag{5.3}$$

$$\sigma(1 - \mu\sigma)\, v^2 k + \frac{d\Pi}{dn} = 0 \tag{5.4}$$

$$\frac{d\Pi}{db} = 0, \tag{5.5}$$

where now,

$$\Pi = p + \tfrac{1}{2}\,\mu\sigma^2 v^2. \tag{5.6}$$

For a Prim gas the equation of state now becomes

$$\sigma = P(p)\, S(\zeta), \tag{5.7}$$

where

$$S(\zeta) = \alpha^2 S(\eta). \tag{5.8}$$

The local sound speed c is defined by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_\eta . \tag{5.9}$$

By use of the relations (2.6), (5.7)-(5.9) we find that

$$c^2 = \frac{1}{P'(p)\, S(\eta)} = \frac{\alpha^2}{P'(p)\, S(\zeta)} = \alpha^2 c_*^2 , \tag{5.10}$$

where c_* is the reduced sound speed. Since the Mach number M is defined by

$$M = \frac{u}{c} ,$$

we find by use of the first equation of (5.1) and (5.10), that

$$M = \frac{v}{c_*} = M_* . \tag{5.11}$$

Hence we see that the Mach number remains invariant under the substitution principle [4]. From (5.8) we find that if F_* is isentropic then F is non-isentropic. It is known that F is complex-lamellar; that is, u_i are orthogonal to a one parameter family of surfaces, if and only if F_* is complex-lamellar ([9], p. 417).

By use of the identity

$$e_{ijk}e^{jlm}\,\partial_l H_m H^k = H^j\,\partial_j H_i - \tfrac{1}{2}\,\partial_i H^2,$$

the equation (2.2) can be written in the form

$$\rho u^j\,\partial_j u_i + \partial_i p + \mu e_{ijk}e^{jlm}\,\partial_l H_m H^k = 0.$$

By substituting for H_i from (3.3) and then for u_i and ρ from (5.1), the above relation becomes

$$v^j\,\partial_j v_i + \frac{1}{\sigma}\,\partial_i p + \mu e_{ijk}e^{jlm}\,\partial_l H_m H^k = 0. \tag{5.12}$$

A flow is barotropic in F_* if σ is a function of p [8]:

$$\sigma = \sigma(p).$$

For such a flow the Bernoulli function q_* exists and is defined by (see (4.9), (5.1))

$$q_*^2 = \frac{1}{2}\,v^2 + \int \frac{dp}{\sigma}.$$

Expressing the equation (5.12) in terms of s_i, n_i and b_i we find that

$$\begin{aligned}\partial_i q_*^2 = &- \left[v\,\frac{dv}{db} + \mu v^2\,\frac{d\sigma}{db} + \mu v\sigma\,\frac{dv}{db}\right] b_i \\ &+ v\left[\mu k\sigma v - \mu\sigma\,\frac{dv}{dn} - \mu v\,\frac{d\sigma}{dn} + kv - \frac{dv}{dn}\right] n_i . \end{aligned} \tag{5.13}$$

From the above result we see that for a barotropic flow: (1) the Bernoulli surfaces $q_* =$ constant contain streamlines; (2) the streamlines are geodesics on the Bernoulli surface if $\partial_i q_*$ is parallel to n_i; (3) if $v(1 + \mu\sigma)$ is constant along the binormal vector then the streamlines form geodesics on the Bernoulli surfaces. Therefore we have the following results: *For a barotropic flow, the Bernoulli surfaces contain the streamlines. Moreover, the streamlines form* *geodesics on the Bernoulli surfaces if and only if* $v(1 + \mu\sigma)$ *is constant along the binormal vector.*

Now we shall answer the question: under what conditions the Bernoulli surfaces contain the vortex lines also? We know that [5]

$$w^i = v\Omega s^i + \frac{dv}{db} n^i + \left(kv - \frac{dv}{dn}\right) b^i, \tag{5.14}$$

where $\Omega = s_i e^{ijk} \partial_j s_k$ the abnormality of the velocity field v_i. From (5.13) and (5.14) we find that

$$w^i \partial_i q_*^2 = \frac{dv}{db} \left[\left(kv - \frac{dv}{dn}\right)(1 + \mu\sigma) - \mu v \frac{d\sigma}{dn}\right] - \left(kv - \frac{dv}{dn}\right)\left(v \frac{dv}{db} + \mu v^2 \frac{d\sigma}{db} + \mu v\sigma \frac{dv}{db}\right). \tag{5.15}$$

The vanishing of the right-hand side of (5.15) is a necessary and sufficient condition for the Bernoulli surfaces to contain the vortex lines. *In particular, we find if the streamlines are geodesics on the Bernoulli surfaces and* $dv/db = 0$, *then the Bernoulli surfaces contain the vortex lines.*

The equation (5.2), by use of (5.7), becomes

$$\frac{d}{ds}\left(\frac{1}{2} v^2\right) + \frac{v^2 P'}{P} \frac{dp}{ds} = - \bar{M} v^2. \tag{5.16}$$

By substituting for Π in (5.3) from (5.6) and cancelling terms we find that (5.3) becomes

$$\frac{d}{ds}\left(\frac{1}{2} v^2\right) = - \frac{1}{SP} \frac{dp}{ds}. \tag{5.17}$$

Eliminating $d(v^2)/ds$ from (5.16) and (5.17) we find that

$$(1 - P' v^2 S) \frac{1}{P} \frac{dp}{ds} = - \bar{M} v^2 S. \tag{5.18}$$

In particular if $\bar{M} = \partial_i s^i = 0$, that is, if the velocity is a family of parallel vector field ([14], p. 258) then (5.18) implies that either

(i) the pressure does not vary along the streamlines

or

(ii) $v^2 P' S = 1$.

Differentiating the condition (ii) along the streamline and making use of (5.17), we find that

$$P(p) = \frac{1}{B - Dp}, \tag{5.19}$$

where B and D are arbitrary constants. Comparing (ii) with (5.10) we find that $v = c_*$ or $u = c$.

Therefore we have the theorem: *if the divergence of the unit tangent vector to streamlines vanishes then either the pressure does not vary along the streamlines, or the flow is sonic.*

The above result holds for nonmagnetic flows [10].

6. Helical Flows

We shall now determine a family of helical flows, which furnish an extension of the family discussed by Wasserman [11] in the nonmagnetic case. The streamlines are along the helices of coaxal cylinders. Such flows have the property that $\bar{M} = \partial_i s^i = 0$ ([14], p. 258). We will introduce the cylindrical coordinates r, θ, z and write

$$s_i = \theta_i \sin\beta + z_i \cos\beta, \tag{6.0}$$

where θ_i and z_i are the unit vectors along the increasing direction of θ and z directions respectively. The angle β of the helices is, in general, a function of r. The helices form geodesics on the cylinders $r =$ constant. With this property, and by use of the Euler's equation for the normal curvature of a curve we find that ([14], p. 73).

$$k = \frac{\sin^2\beta}{r}.$$

We also have that

$$b_i = -\theta_i \cos\beta + z_i \sin\beta, \qquad n_i = -r_i,$$

where r_i is the unit vector along the radius of the cylinder. With the help of the above relations, (5.2)-(5.5) reduce to

$$\frac{d}{ds}(\sigma v) = 0, \tag{6.1}$$

$$\sigma v \frac{dv}{ds} = -\frac{dp}{ds}, \tag{6.2}$$

$$\sigma(1 - \mu\sigma)\, v^2 \frac{\sin^2\beta}{r} = \frac{\partial\Pi}{\partial r}, \tag{6.3}$$

$$\frac{d\Pi}{db} = 0, \tag{6.4}$$

where

$$\frac{d}{ds} = \frac{\sin\beta}{r}\frac{\partial}{\partial\theta} + \cos\beta\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{d}{db} = -\frac{\cos\beta}{r}\frac{\partial}{\partial\theta} + \sin\beta\frac{\partial}{\partial z}.$$

The equations

$$\sigma = P(p)\, S(\zeta), \qquad \frac{d\zeta}{ds} = 0 \tag{6.5}$$

must be added to the system. Since $\bar{M} = 0$, we find from the last part of Section 5 that either the pressure does not vary along the streamlines or the fluid flows with sonic speed; that is, either

(a) $dp/ds = 0$

or

(b) $u = c$.

For flows with (a) we find from (6.2) and (6.5) that

$$\frac{d\sigma}{ds} = \frac{dv}{ds} = 0,$$

and therefore the system reduces to

$$\frac{d\Pi}{dr} = \sigma(1 - \mu\sigma)\, v^2 \frac{\sin^2\beta}{r}. \tag{6.6}$$

In the above equation it is understood that σ and v are constant along the streamlines. Π is a nondecreasing function of r if $\sigma\mu < 1$, attains a maximum if $\sigma\mu = 1$, and is nonincreasing if $\sigma\mu > 1$. In the nonmagnetic case, the fluid pressure p is a nondecreasing function of r [11]. It is easy to construct examples with p bounded or unbounded; σ, v and β may, in general, either increase or decrease with r; σ and v may vary from one streamline to another as long as $\sigma(1 - \mu\sigma)\, v^2$ remains constant on each cylinder.

We shall now consider an example in which p decreases with r. Let us assume that the flow F_* is nonisentropic flow of a polytropic gas. If we assume that σ, p, v, $S(\zeta)$, $(\sin^2\beta)/r$ are of the form Ae^{nr}, where A and n are constants, and substitute into the equation (6.6), we find that one set of solutions is

$$\sigma = Ae^{-tr}, \qquad p = \frac{\mu}{m} e^{-mr}, \qquad v^2 = \frac{1}{A^2} e^{(3t-m)r}$$

$$S(\zeta) = A\left(\frac{m}{\mu}\right)^{1/\gamma} e^{-(d/\gamma)r}, \qquad \frac{\sin^2\beta}{r} = e^{-tr}, \tag{6.7}$$

where

$$m = \gamma t - d, \qquad A = \frac{2}{\mu(t-m)}, \qquad t > m > 0, \qquad d > 0,$$

and γ is the adiabatic exponent. From (5.10) we find that the local sound speed is given by

$$c_*^2 = \frac{\mu}{mA} e^{(t-m)r},$$

and therefore the Mach number is given by the equation

$$M^2 = \frac{v^2}{c_*^2} = \frac{1}{2(t-m)} e^{2tr}.$$

All the flow quantities are bounded at the origin, and as r approaches infinity, p, σ, $S(\zeta)$ and Π decrease to zero and v, c_*, M increase to infinity; β increases from $r = 0$ to $r = 1$ and decreases to 0 steadily as r approaches infinity. The vorticity vector is found to be

$$w_*^i = v^2 \left[\frac{d\beta}{dr} + \frac{\sin\beta\cos\beta}{r}\right] s^i + \left[v\frac{\sin^2\beta}{r} + \frac{dv}{dr}\right] b^i,$$

where v and β are given by (6.7). After substituting for v and β in the above equation we find that the motion is irrotational. Since α is a function that is constant along the streamlines,

$$\frac{\sin\beta}{r}\frac{\partial\alpha}{\partial\theta} + \cos\beta\frac{\partial\alpha}{\partial z} = 0,$$

whose solution is

$$\alpha = g(z - r\theta\cot\beta)\, h(r), \tag{6.8}$$

where h and g are arbitrary functions of r and $z - r\theta \cot\beta$ respectively. From (5.1) the flow F, corresponding to F_* given by the equation (6.7), has the following flow parameters:

$$u = \alpha v, \qquad \rho = \frac{\sigma}{\alpha^2}, \qquad p = p, \qquad s(\eta) = \frac{s(\zeta)}{\alpha^2}, \tag{6.9}$$

where v, σ, p, $s(\zeta)$ are given by (6.7) and α is given by (6.8). Therefore the velocity vector and the magnetic field vector, from (3.3), (4.1), (6.0), (6.7), and (6.8) are

$$u_i = \frac{\alpha}{A} e^{1/2(3t-m)r}(\theta_i \sin\beta + z_i \cos\beta) = \frac{H_i}{\alpha\rho}. \tag{6.10}$$

And ρ, $S(\eta)$ and p are given by (6.9) and (6.7):

$$\rho = \frac{1}{\alpha^2} Ae^{-tr}, \qquad S(\eta) = \frac{1}{\alpha^2} A \left(\frac{m}{\mu}\right)^{1/\gamma} e^{-(d/\gamma)r}, \qquad p = \frac{\mu}{m} e^{-mr}. \tag{6.11}$$

In the above equation α is given by (6.8) and β is given by the fifth equation of (6.7).

When the equations of continuity, motion and Maxwell's relations (2.1)-(2.5) are transformed into polar coordinates, we find that the velocity and field components given by (6.10) and ρ, $S(\eta)$ and p of (6.11) identically satisfy the transformed equations.

Now we shall investigate the case (b). Introducing the transformation [11].

$$\phi = z + \theta r \tan\beta,$$
$$\psi = z - \theta r \cot\beta,$$
$$r = r,$$

the equation (6.3) becomes by use of (5.19),

$$r\frac{\partial \Pi}{\partial r} + (\phi - \psi)\sin\beta\cos\beta \frac{d}{dr}(\tan\beta)\frac{\partial \Pi}{\partial \phi} = (R - p)\sin^2\beta, \tag{6.12}$$

where R is a constant. A number of flows could be found to satisfy (6.12).

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References

1. H. Grad. Reducible problems in magneto-fluid dynamics steady flows. *Rev. Modern Phys.* 32 (1960), 830-847.
2. P. Germain. Sur Certains Écoulements D'un Fluide Parfaitement Conducteur. La Recherche Aéronautique No. 74, Janvier-Février 1960.
3. P. Smith. The steady magnetodynamic flow of perfectly conducting fluids. *J. Math. Mech.* 12 (1963), 505-520.
4. R. C. Prim. Steady rotational flow of ideal gases. *J. Rational Mech. Anal.* 1 (1952), 425-297.
5. N. Coburn. Intrinsic relations satisfied by the vorticity and velocity vectors in fluid flow theory. *Michigan Math. J.* (1952), 113-130.

6. E. R. SURYANARAYAN. On the geometry of streamlines in hydromagnetic fluid flows when the magnetic field is along a fixed direction. *Proc. Amer. Math. Soc.* **16** (1965), 90-96.
7. P. SMITH. Some intrinsic properties of spatial gas flows. *J. Math. Mech.* 12 (1963), 27-32.
8. J. SERRIN. Mathematical principles of classical fluid mechanics. *Handbuch Physik* **VII**/1 (1959), 150.
9. C. TRUESDELL AND R. TOUPIN. Principles of classical mechanics and field theory. *Handbuch Physik* **III**/1 (1960).
10. R. H. WASSERMAN. Formulations and solutions of the equations of fluid flow. Unpublished thesis, University of Michigan, 1957.
11. R. H. WASSERMAN. Helical fluid flow. *Quart. Appl. Math.* **17** (1960), 443-445.
12. R. H. WASSERMAN. A class of three-dimensional compressible fluid flows. *J. Math. Anal. Appl.* **5** (1962), 119-135.
13. K. O. FRIEDERICHS AND H. KRANZER. Non-linear wave motion, *A. E. C. Res. Develop. Rept.* NYO-6486 (1958), 11.
14. C. E. WEATHERBURN. "Differential Geometry of Three Dimensions." Vol. I, Cambridge University Press, 1955.